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MAGNETIC RIGIDITY OF HOROCYCLE FLOWS

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Abstract. Let M be a closed oriented surface endowed with a Riemannian metric g and let Ω be a 2-form. We show that the magnetic flow of the pair (g,Ω) has zero asymptotic Maslov index and zero Liouville action if and only g has constant Gaussian curvature, Ω is a constant multiple of the area form of g and the magnetic flow is a horocycle flow.

This characterization of horocycle flows implies that if the magnetic flow of a pair (g,Ω) is C^1-conjugate to the horocycle flow of a hyperbolic metric $\bar{g}$ then there exists a constant $a>0$, such that ag and $\bar{g}$ are isometric and $a^{-1}\Omega$ is, up to a sign, the area form of g. The characterization also implies that if a magnetic flow is Mañé critical and uniquely ergodic it must be the horocycle flow.

As a by-product we also obtain results on existence of closed magnetic geodesics for almost all energy levels in the case weakly exact magnetic fields on arbitrary manifolds.

1. Introduction

Let Γ be a cocompact lattice of $PSL(2,\mathbb{R})$. The standard horocycle flow h is given by the right action of the one-parameter subgroup

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

on $\Gamma\backslash PSL(2,\mathbb{R})$. The horocycle flow is known to display very peculiar ergodic properties. It preserves the Riemannian volume on $\Gamma\backslash PSL(2,\mathbb{R})$, is uniquely ergodic [18], and mixing of all degrees [29]. Morever, it has zero entropy since

$$\phi^0_t \circ h_s = h_{se^{-t}} \circ \phi^0_t \tag{1}$$

for all $s,t\in\mathbb{R}$, where ϕ^0 is the geodesic flow given by the one-parameter subgroup:

$$\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}.$$

In fact, h parametrizes the strong stable manifold of ϕ^0. The horocycle flow is a very rigid object, as the works of B. Marcus and M. Ratner show [30, 34, 35]. Recent results on ergodic averages and solutions of cohomological equations for h can be found in [4, 17].

In the present paper we would like to look at horocycle flows as magnetic flows. A matrix X in $sl(2,\mathbb{R})$ gives rise to a flow ϕ on $\Gamma\backslash PSL(2,\mathbb{R})$ by setting

$$\phi_t(\Gamma g) = \Gamma g e^{tX}.$$

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The geodesic and horocycle flows are just particular cases of these algebraic flows. Consider the following path of matrices in $sl(2,\mathbb{R})$:

$$\mathbb{R} \ni \lambda \mapsto X_\lambda := \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}.$$

The flows ϕ^λ on $\Gamma\backslash PSL(2,\mathbb{R})$ associated with the matrices X_λ have an interesting interpretation. Since $PSL(2,\mathbb{R})$ acts by isometries on the hyperbolic plane $\mathbb{H}^2$, $M := \Gamma\backslash\mathbb{H}^2$ is a compact hyperbolic surface (provided Γ acts without fixed points) and the unit sphere bundle SM of M can be identified with $\Gamma\backslash PSL(2,\mathbb{R})$. A calculation shows that ϕ^λ is the Hamiltonian flow of the Hamiltonian $H(x,v) = \frac{1}{2}|v|_x^2$ with respect to the symplectic form on TM given by

$$-d\alpha + \lambda\, \pi^*\Omega_a,$$

where Ω_a is the area form of M, $\pi : TM \to M$ is the canonical projection and α is the contact 1-form that generates the geodesic flow of M. For $\lambda = 0$, ϕ^0 is the geodesic flow and for $\lambda = 1$, ϕ^1 is the flow induced by the one-parameter subgroup with matrix on $sl(2,\mathbb{R})$ given by

$$X_1 = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{pmatrix}.$$

Since there exists an element $c \in PSL(2,\mathbb{R})$ such that

$$c^{-1}X_1 c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

the map $f : \Gamma\backslash PSL(2,\mathbb{R}) \to \Gamma\backslash PSL(2,\mathbb{R})$ given by $f(\Gamma g) = \Gamma g c$ conjugates ϕ^1 and h, i.e. $f \circ \phi^1_t = h_t \circ f$. In fact, any matrix in $sl(2,\mathbb{R})$ with determinant zero will give rise to a flow which is conjugate to h_t or h_{-t}. (So, up to orientation, there is just one algebraic horocycle flow.) Passing by, we note that $\det X_\lambda = -\frac{1}{4}(1-\lambda^2)$, so for $|\lambda| < 1$, the flow ϕ^λ is conjugate to the geodesic flow ϕ^0, up to a constant time scaling by $\sqrt{1-\lambda^2}$. Hence the magnetic flows ϕ^λ are just geodesic flows, but with entropy $\sqrt{1-\lambda^2}$. (This latter observation is due to V.I. Arnold [2].)

In general, if (M,g) is a closed Riemannian manifold and Ω is a closed 2-form, the Hamiltonian flow ϕ of $H(x,v) = \frac{1}{2}|v|_x^2$ with respect to the symplectic form on TM given by

$$\omega := -d\alpha + \pi^*\Omega,$$

is called the *magnetic flow* of the pair (g,Ω) because it models the motion of a particle under the influence of the magnetic field Ω. The projection of the orbits of ϕ to M are called *magnetic geodesics*. The discussion above shows that h appears as the magnetic flow of a hyperbolic surface with Ω equal to the area form of the surface.

Since the horocycle flow has no closed orbits, this already gives an example of a Hamiltonian system with an energy level (SM) without closed orbits. This example has been much used, most notably by V. Ginzburg [19, 20, 21] to give smooth counterexamples to the Hamiltonian Seifert conjecture in $\mathbb{R}^{2n}$, $n \geq 3$ (a C^2-counterexample is now available in $\mathbb{R}^4$ [22]).

How frequently does the horocycle flow appear as a magnetic flow? To answer this question we first prove a characterization of horocycle flows within the set of magnetic flows.

Magnetic flows on surfaces leave invariant the volume form $\alpha \wedge d\alpha$. The associated Borel probability measure is called the *Liouville measure* μ_ℓ of SM. We shall assume from now on that M has genus ≥ 2. Then $\pi^* : H^2(M, \mathbb{R}) \to H^2(SM, \mathbb{R})$ is the zero map and thus if Ω is any 2-form, $\pi^*\Omega$ is exact on SM. It follows that ω restricted to SM is exact and we let Θ be any primitive. We define *the action of the Liouville measure* as:

$$\mathfrak{a}(\mu_\ell) := \int \Theta(X)\, d\mu_\ell,$$

where X is the vector field on SM that generates the magnetic flow. The action does not depend on the primitive, since the *asymptotic cycle* of μ_ℓ is zero (cf. Section 2 and [9]), i.e. for any closed 1-form φ on SM we have

$$\int \varphi(X)\, d\mu_\ell = 0.$$

It is quite simple to check that when M is a hyperbolic surface and Ω is the area form, $\mathfrak{a}(\mu_\ell) = 0$. In fact, in this case, there is a primitive Θ, with $\Theta(X) \equiv 0$. It is also easy to check that there are no conjugate points [10][Example A.1]. Equivalently, using the results in [9], we can say that *the asymptotic Maslov index* $\mathfrak{m}(\mu_\ell)$ of the Liouville measure is zero (cf. Section 2).

We first show that these two symplectic-ergodic quantities characterize horocycle flows.

Proposition. *Let M be a closed oriented surface endowed with a Riemannian metric g and let Ω be a 2-form. The magnetic flow of the pair (g, Ω) has $\mathfrak{a}(\mu_\ell) = \mathfrak{m}(\mu_\ell) = 0$ if and only g has constant Gaussian curvature, Ω is a constant multiple of the area form of g and the magnetic flow is a horocycle flow.*

The Proposition has the following consequence:

Theorem A. *Let M be a closed oriented surface endowed with a Riemannian metric g and let Ω be a 2-form. If the magnetic flow of the pair (g, Ω) is C^1-conjugate to the horocycle flow of a hyperbolic metric $\bar{g}$, there exists a constant $a > 0$, such that ag and $\bar{g}$ are isometric and $a^{-1}\Omega$ is, up to a sign, the area form of g.*

We observe that it is not possible to conclude that g has curvature -1 (i.e. $a = 1$). This is simply because the magnetic flow of a pair (g, Ω) with g of constant negative curvature $-k$, $\Omega = \lambda\, \Omega_a$ ($\lambda > 0$) and $\lambda^2 = k$ is smoothly conjugate to the horocycle flow of a hyperbolic surface. To see this observe that an easy scaling argument shows that the magnetic flow of a such pair is, up to a constant time change, conjugate to the horocycle flow of a hyperbolic surface. But by (1), h_t is conjugate to $h_{\tau t}$ for any positive real number τ. This shows that the area A of a surface is *not* preserved under C^1-conjugacies of magnetic flows. C. Croke and B. Kleiner [13] have shown that the volume of a Riemannian manifold is preserved under C^1-conjugacies of geodesic

flows. In the case of transitive magnetic flows, we show that C^1-conjugacies preserve $\mathfrak{a}(\mu_\ell)/A$ (cf. Lemma 4.1).

We now describe a second and more involved application of the Proposition. Let $\widetilde{\Omega}$ be the lift of Ω to the universal cover $\widetilde{M} \cong \mathbb{R}^2$ of M. Since $\widetilde{\Omega}$ is an exact form, there exists a smooth 1-form θ such that $\widetilde{\Omega} = d\theta$. Let us consider the Lagrangian on $\widetilde{M}$ given by

$$L(x, v) = \frac{1}{2}|v|_x^2 - \theta_x(v).$$

It is well known that the extremals of L, i.e., the solutions of the Euler-Lagrange equations of L,

$$\frac{d}{dt}\frac{\partial L}{\partial v}(x, v) = \frac{\partial L}{\partial x}(x, v)$$

coincide with the lift to $\widetilde{M}$ of the magnetic geodesics. The action of the Lagrangian L on an absolutely continuous curve $\gamma : [a, b] \to \widetilde{M}$ is defined by

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt.$$

The *Mañé critical value* of the pair (g, Ω) is

$$c = c(g, \Omega) := \inf\{k \in \mathbb{R} :\ A_{L+k}(\gamma) \geq 0 \text{ for any absolutely continuous closed curve } \gamma \text{ defined on any closed interval } [0, T]\ \}.$$

Like any Lagrangian flow, the magnetic flow for $T\widetilde{M}$ can be viewed as the Hamiltonian flow defined by the canonical symplectic form on $T^*\widetilde{M}$ and a suitable Hamiltonian function $H : T^*\widetilde{M} \to \mathbb{R}$; in this case

$$H(x, p) = \frac{1}{2}|p + \theta_x|^2.$$

The Legendre transform $\mathcal{L} : T\widetilde{M} \to T^*\widetilde{M}$ defined by

$$\mathcal{L}(x, v) = \frac{\partial L}{\partial v}(x, v)$$

carries orbits of the Lagrangian flow for L to orbits of the Hamiltonian flow defined by H and the canonical symplectic form. The critical value can also be defined in Hamiltonian terms [5] as:

$$\begin{aligned} c(g, \Omega) &= \inf_{u \in C^\infty(\widetilde{M}, \mathbb{R})} \sup_{x \in \widetilde{M}} H(x, d_x u) \\ &= \inf_{u \in C^\infty(\widetilde{M}, \mathbb{R})} \sup_{x \in \widetilde{M}} \frac{1}{2}|d_x u + \theta_x|^2. \end{aligned}$$

As u ranges over $C^\infty(\widetilde{M}, \mathbb{R})$ the form $\theta - du$ ranges over all primitives of $\widetilde{\Omega}$, because any two primitives differ by a closed 1-form which must be exact since $\widetilde{M}$ is simply connected. Since on a surface of genus ≥ 2, there are bounded primitives we always have $c(g, \Omega) < \infty$.

We will say that a magnetic flow is *Mañé critical* if $c(g, \Omega) = 1/2$. Magnetic flows which are supercritical, i.e. $1/2 > c(g, \Omega)$ always have positive topological entropy [5][Proposition 5.4]. Hence such flows exhibit a horseshoe and exponential growth rate of hyperbolic closed magnetic geodesics. In fact, we will show that, in any dimension, if $1/2 > c(g, \Omega)$, then a nontrivial homotopy class contains a closed magnetic geodesic provided that the centralizer of some element in the class is an amenable subgroup (cf. Theorem 5.5). For subcritical magnetic flows, i.e. $1/2 < c(g, \Omega)$ one hopes to prove that there are always closed contractible magnetic geodesics, although nothing of this kind has been proved in general. When Ω itself is exact, the main result in [12] says that there always exists a closed magnetic geodesic.

What happens for magnetic flows which are Mañé critical with Ω non-exact? It is easy to check that the hororcycle flow is Mañé critical (cf. [6][Example 6.2]). Is it the only magnetic flow which is Mañé critical and uniquely ergodic? Aubry-Mather theory combined with the Proposition gives the answer:

Theorem B. *Let M be a closed oriented surface endowed with a Riemannian metric g and let Ω be a 2-form. Suppose the magnetic flow of the pair (g, Ω) is Mañé critical and uniquely ergodic. Then g has constant Gaussian curvature, Ω is a constant multiple of the area form of g and the magnetic flow is a horocycle flow.*

Our results on the existence of closed magnetic geodesics in non-trivial free homotopy classes for supercritical magnetic flows combined with recent results of G. Contreras [7] and O. Osuna [31] imply the following statement on almost existence of closed magnetic geodesics for weakly exact magnetic flows. Recall that Ω is said to be *weakly exact* if its lift to the universal covering of M is exact.

Theorem C. *Let M be an arbitrary closed manifold endowed with a Riemannian metric g and let Ω be a weakly exact 2-form. We have:*

1. *if $\pi_1(M)$ is amenable and Ω is not exact, then almost every energy level contains a contractible closed magnetic geodesic;*
2. *if $\pi_1(M)$ contains a non-trivial element with an amenable centralizer, then almost every energy level contains a closed magnetic geodesic.*

Recall that a discrete group Γ is said to be *amenable* if the space of bounded functions $\Gamma \to \mathbb{R}$ has a left (or right) invariant mean [33]. Examples are finite groups, abelian groups and finite extensions of solvable groups. If a group contains a free subgroup on two generators, then is non-amenable. I do not know of any example of a finitely presented group, for which the centralizer of every element is non-amenable. We note that Contreras in [7] proved item 2 in Theorem C when Ω is exact without any assumption on $\pi_1(M)$ and V. Ginzburg and E. Kerman proved item 2 when M is a torus [23].

There are several results establishing the existence of closed contractible magnetic geodesics for almost every *low* energy level, see [36, 27] and references therein. The methods of Symplectic Topology have proven to be effective in this respect. For high energy levels, at least when g is generic, one can also obtain existence of closed magnetic geodesics in free homotopy classes by observing that magnetic flows approach

geodesics flows as energy increases. Very little is known about how to bridge the gap. The exceptions seem to be Theorem C and the main result in [12] alluded above, which are both based on Aubry-Mather theory.

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2. Geometric preliminaries

Let M be a closed oriented surface, SM the unit sphere bundle and $\pi : SM \to M$ the canonical projection. The latter is in fact a principal S^1-fibration and we let V be the infinitesimal generator of the action of S^1.

Given a unit vector $v \in T_xM$, we will denote by iv the unique unit vector orthogonal to v such that $\{v, iv\}$ is an oriented basis of T_xM. There are two basic 1-forms α and β on SM which are defined by the formulas:

$$\alpha_{(x,v)}(\xi) := \langle d_{(x,v)}\pi(\xi), v\rangle;$$
$$\beta_{(x,v)}(\xi) := \langle d_{(x,v)}\pi(\xi), iv\rangle.$$

The form α is precisely the contact form that we mentioned in the introduction. The vector field X_0 uniquely determined by the equations $\alpha(X_0) \equiv 1$, $i_{X_0}d\alpha \equiv 0$ generates the geodesic flow ϕ^0 of M.

A basic theorem in 2-dimensional Riemannian geometry asserts that there exists a unique 1-form ψ on SM (the connection form) such that $\psi(V) \equiv 1$ and

$$\begin{aligned} d\alpha &= \psi \wedge \beta \\ d\beta &= -\psi \wedge \alpha \\ d\psi &= -(K \circ \pi)\, \alpha \wedge \beta \end{aligned}$$

where K is the Gaussian curvature of M. In fact, the form ψ is given by

$$\psi_{(x,v)}(\xi) = \left\langle \frac{DZ}{dt}(0), iv \right\rangle,$$

where $Z : (-\varepsilon, \epsilon) \to SM$ is any curve with $Z(0) = (x, v)$ and $\dot{Z}(0) = \xi$ and $\frac{DZ}{dt}$ is the covariant derivative of Z along the curve $\pi \circ Z$.

It is easy to check that $\alpha \wedge \beta = \pi^*\Omega_a$, hence

$$d\psi = -\pi^*(K\,\Omega_a). \tag{2}$$

In the case of a hyperbolic surface, the vertical vector field V corresponds to the matrix in $sl(2, \mathbb{R})$ given by

$$\begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}.$$

2.1. **Asymptotic cycles.** Given a ϕ-invariant Borel probability measure μ, *the asymptotic cycle* of μ is the real 1-homology class $\mathcal{S}(\mu)$ defined by the equality:

$$\langle [\varphi], \mathcal{S}(\mu) \rangle = \int_{SM} \varphi(X)\, d\mu$$

for any closed 1-form φ. Let us check that $\mathcal{S}(\mu_\ell) = 0$. We will prove something slightly more general that we will need later. Let Λ_0^* be the space of continuous forms λ whose exterior derivative, weakly defined by Stokes' theorem: $\int_\sigma d\lambda = \int_{\partial\sigma} \lambda$ for every smooth chain σ, are also continuous differential forms. The space Λ_0^* is closed under exterior differentiation, wedge products and pull back of C^1 maps.

Let φ be a continuous 1-form which is closed in the sense that its integral over the boundary of any 2-chain is zero, i.e. $\varphi \in \Lambda_0^1$ and $d\varphi = 0$. We claim that

$$\int_{SM} \varphi(X)\, d\mu_\ell = 0.$$

Note that we always have $\alpha \wedge \pi^*\Omega = 0$ as it easily follows from evaluating the 3-form on any basis that contains V. Thus $\alpha \wedge (-d\alpha) = \alpha \wedge \omega$ and it suffices to show that

$$\int_{SM} \varphi(X)\, \alpha \wedge \omega = 0.$$

Observe that

$$\varphi(X)\, \alpha \wedge \omega = \varphi \wedge i_X(\alpha \wedge \omega) = \varphi \wedge \omega,$$

since $i_X\omega = 0$. But since ω is exact, if we let Θ be a primitive, we have $d(\varphi \wedge \Theta) = \varphi \wedge \omega$ and the claim follows from our definition of exterior differentiation and the fact that M is a closed surface.

2.2. **Asymptotic Maslov index.** Let $\Lambda(SM)$ be the set of Lagrangian subspaces of $T|_{SM}TM$. Given any subspace $E \in \Lambda(SM)$ and $T > 0$ we can consider the curve of Lagrangian subspaces $[0, T] \ni t \mapsto d\phi_t(E)$. Let $n(x, v, E, T)$ be the intersection number of this curve with the Maslov cycle of $\Lambda(SM)$. It was shown in [9] that if μ is ϕ-invariant, the limit

$$\mathfrak{m}(x, v) := \lim_{T\to\infty} \frac{1}{T} n(x, v, E, T)$$

exists for μ-almost every (x, v), is independent of E, and $(x, v) \mapsto \mathfrak{m}(x, v)$ is integrable. *The asymptotic Maslov index* of μ is:

$$\mathfrak{m}(\mu) := \int_{SM} \mathfrak{m}(x, v)\, d\mu(x, v).$$

2.3. **Green subbundles.** Given $(x, v) \in TM$ we define *the vertical subspace at* (x, v) as $\mathcal{V}(x, v) := \ker d_{(x,v)}\pi$, where $\pi : TM \to M$ is the canonical projection. Note that $\mathcal{V}(x, v) \cap T_{(x,v)}SM$ is spanned by the value of the vector field V at (x, v). We say that the orbit of $(x, v) \in SM$ *does not have conjugate points* if for all $t \neq 0$,

$$d_{(x,v)}\phi_t(\mathcal{V}(x, v)) \cap \mathcal{V}(\phi_t(x, v)) = \{0\}.$$

Since magnetic flows are optical, the main result in [9] says that SM has no conjugate points (i.e. for all $(x,v) \in SM$, the orbit of (x,v) does not have conjugate points) if and only if $\mathfrak{m}(\mu_\ell) = 0$.

If SM has no conjugate points, one can construct the so called *Green subbundles* [10, Proposition A] given by:

$$E(x,v) := \lim_{t\to+\infty} d\phi_{-t}(\mathcal{V}(\phi_t(x,v))),$$

$$F(x,v) := \lim_{t\to+\infty} d\phi_{t}(\mathcal{V}(\phi_{-t}(x,v))).$$

These subbundles are Lagrangian, they never intersect the vertical subspace and, crucial for us, they are contained in $T(SM)$. Moreover, they vary measurably with (x,v) and they contain the vector field X.

3. Proof of the Proposition

The proof will be based on integrating an appropriate Riccati equation along a solution arising from the Green bundles. This is a well known method, first used by E. Hopf [26] and subsequently extended to higher dimensions by L.W. Green [25]. The method is still paying dividends, cf. [3, 24].

Let Ω be an arbitrary smooth 2-form. We write $\Omega = f\,\Omega_a$, where $f : M \to \mathbb{R}$ is a smooth function and Ω_a is the area form of g. Let A be the area of g.

Since $H^2(M,\mathbb{R}) = \mathbb{R}$, there exist a constant c and a smooth 1-form ϱ such that

$$\Omega = cK\,\Omega_a + d\varrho$$

and $c = 0$ if and only if Ω is exact. Using (2) we have

$$\omega := -d\alpha + \pi^*\Omega = d(-\alpha - c\,\psi + \pi^*\varrho).$$

The vector field X that generates the magnetic flow ϕ is given by $X = X_0 + f\,V$ since it satisfies the equation $dH = i_X\omega$. Since X_0 and V preserve the volume form $\alpha \wedge d\alpha$, then so does $X = X_0 + fV$ and thus ϕ preserves the normalized Liouville measure μ_ℓ of SM.

If we evaluate the primitive $-\alpha - c\,\psi + \pi^*\varrho$ of the symplectic form ω on X we obtain:

$$(-\alpha - c\,\psi + \pi^*\varrho)(X)(x,v) = -1 - c\,f(x) + \varrho_x(v). \tag{3}$$

Therefore $\mathfrak{a}(\mu_\ell)$ is given by:

$$\mathfrak{a}(\mu_\ell) = -1 - \frac{c}{A}\int_M f(x)\,dx$$

since μ_ℓ is invariant under the flip $v \mapsto -v$. By the definition of c and f and the Gauss-Bonnet theorem

$$\int_M f(x)\,dx = c\int_M K(x)\,dx = 2\pi\,\chi\,c$$

and hence

$$\mathfrak{a}(\mu_\ell) = -1 - \frac{1}{2\pi\chi A}\left(\int_M f(x)\,dx\right)^2. \tag{4}$$

Given $(x,v) \in SM$ and $\xi \in T_{(x,v)}TM$, let

$$J_\xi(t) = d_{(x,v)}(\pi \circ \phi_t)(\xi).$$

We call J_ξ a *magnetic Jacobi field* with initial condition ξ. It was shown in [32] that J_ξ satisfies the following Jacobi equation:

$$\ddot{J}_\xi + R(\dot\gamma, J_\xi)\dot\gamma - [Y(\dot{J}_\xi) + (\nabla_{J_\xi}Y)(\dot\gamma)] = 0, \tag{5}$$

where $\gamma(t) = \pi \circ \phi_t(x,v)$, R is the curvature tensor of g and Y is determined by the equality $\Omega_x(u,v) = \langle Y_x(u), v\rangle$ for all $u, v \in T_xM$ and all $x \in M$.

Let us express J_ξ as follows:

$$J_\xi(t) = x(t)\dot\gamma(t) + y(t)i\dot\gamma(t),$$

and suppose in addition that $\xi \in T_{(x,v)}SM$, which implies

$$g_\gamma(\dot{J}_\xi, \dot\gamma) = 0. \tag{6}$$

A straightforward computation using (5) and (6) shows that x and y must satisfy the scalar equations:

$$\dot{x} = f(\gamma)\,y \tag{7}$$

$$\ddot{y} + \left[K(\gamma) - \langle \nabla f(\gamma), i\dot\gamma\rangle + f^2(\gamma)\right] y = 0. \tag{8}$$

Note that the no conjugate points condition is equivalent to saying that any non-trivial magnetic Jacobi field which vanishes at $t = 0$ is never zero again.

Let us consider one of the Green subbundles, let us say E. Since for any $(x,v) \in SM$ the subspace E does not intersect the vertical subspace $\mathcal{V}(x,v)$, there exists a linear map $S(x,v) : T_xM \to T_xM$ such that E can be identified with the graph of S. Let $u(x,v)$ be the trace of $S(x,v)$. An easy calculation using (7) and (8) shows that u along ϕ satisfies the Riccati equation:

$$\dot{u} + u^2 + K(\gamma) - \langle \nabla f(\gamma), i\dot\gamma\rangle + f^2(\gamma) = 0. \tag{9}$$

We can now integrate equation (9) with respect to $t \in [0,1]$ and then with respect to μ_ℓ (using the ϕ-invariance of μ_ℓ) to conclude that:

$$\int_{SM} u^2\,d\mu_\ell + \int_{SM} [K(x) - \langle \nabla f(x), iv\rangle + f^2(x)]\,\mu_\ell = 0.$$

Since μ_ℓ is invariant under the flip $v \mapsto -v$ we have:

$$\int_{SM} \langle \nabla f(x), iv\rangle\; d\mu_\ell = 0$$

and thus

$$\int_{SM} u^2\,d\mu_\ell + \int_{SM} [K(x) + f^2(x)]\,\mu_\ell = 0.$$

The last equality implies

$$\int_{SM}[K(x)+f^2(x)]\,\mu_\ell=\frac{2\pi\chi}{A}+\frac{1}{A}\int_M f^2(x)\,dx\leq 0 \tag{10}$$

with equality if and only if u is zero for almost every $(x,v)\in SM$. But if we now assume that the action $\mathfrak{a}(\mu_\ell)$ vanishes, equation (4) and the Cauchy-Schwarz inequality tell us that:

$$-2\pi\chi\,A=\left(\int_M f(x)\,dx\right)^2\leq A\int_M f^2(x)\,dx.$$

Combining the last inequality with (10) we see that f must be constant and u is zero for almost every $(x,v)\in SM$. If we now use this information in the Riccati equation (9) we conclude that K must be constant and $K+f^2=0$. The last equality ensures that the magnetic flow is a horocycle flow thus concluding the proof of the Proposition.

4. Proof of Theorem A

Lemma 4.1. *Let M_i, $i=1,2$ be closed oriented surfaces with magnetic flows ϕ^i determined by pairs (g_i,Ω_i), $i=1,2$. Suppose ϕ^1 is C^1-conjugate to ϕ^2 and one of them is transitive. Then*

$$A_2\,\mathfrak{a}(\mu_\ell^1)=A_1\,\mathfrak{a}(\mu_\ell^2),$$

where A_i is the area of g_i.

Proof. Let $f:SM_1\to SM_2$ be the C^1-conjugacy and ω_i the corresponding symplectic forms restricted to SM_i. Recall that $\alpha\wedge(-d\alpha)=\alpha\wedge\omega$. Since magnetic flows preserve $\alpha\wedge d\alpha$, the volume form $f^*(\alpha_2\wedge\omega_2)$ is invariant under ϕ^1. Since we are assuming that the magnetic flows are transitive there exists a (nonzero) constant κ such that

$$f^*(\alpha_2\wedge\omega_2)=\kappa\,\alpha_1\wedge\omega_1. \tag{11}$$

Note that df maps X_1 to X_2, $\alpha_i(X_i)=1$ and $i_{X_i}\omega_i=0$, hence contracting with X_1 in the last equation gives:

$$f^*\omega_2=\kappa\,\omega_1.$$

Let Θ_i be a primitive of ω_i. Then $\varphi:=f^*\Theta_2-\kappa\,\Theta_1$ is a continuous 1-form, which is closed in the sense that its integral over the boundary of every 2-chain is zero. By (11), $f_*\mu_\ell^1=\mu_\ell^2$, thus

$$\int_{SM_1}\varphi(X_1)\,d\mu_\ell^1=\int_{SM_2}\Theta_2(X_2)\,d\mu_\ell^2-\kappa\int_{SM_1}\Theta_1(X_1)\,d\mu_\ell^1=\mathfrak{a}(\mu_\ell^2)-\kappa\,\mathfrak{a}(\mu_\ell^1).$$

But, since the asymptotic cycle of μ_ℓ is zero (cf. Subsection 2.1), the left hand side vanishes. Equality (11) implies that $\kappa=A_2/A_1$ and the lemma follows.

□

Lemma 4.2. *Let M_i, $i=1,2$ be closed oriented surfaces with magnetic flows ϕ^i determined by pairs (g_i,Ω_i), $i=1,2$. Suppose ϕ^1 is C^1-conjugate to ϕ^2 and let $f:SM_1\to SM_2$ be the conjugacy. Then $\mathfrak{m}(\mu_\ell^1)=\mathfrak{m}(f_*\mu_\ell^1)$. In particular, ϕ^1 has conjugate points if and only if ϕ^2 does.*

Proof. Let $W_1(x,v)$ be the subspace of $T_{(x,v)}SM_1$ spanned by the magnetic vector field X_1 and the vertical vector field V. Since W_1 contains the magnetic vector field and it is 2-dimensional it must be a Lagrangian subbundle. Since df maps X_1 to X_2, it also maps Lagrangian subspaces contained in $T(SM_1)$ to Lagrangian subspaces contained in $T(SM_2)$. In particular, the subbundle $W_2 := df(W_1)$ must be a Lagrangian subbundle contained in $T(SM_2)$.

We now invoke the fact [9][Corollary 3.2] that the asymptotic Maslov index of a measure with zero asymptotic cycle does not depend on the continuous Lagrangian section that is used to compute it. Thus we can compute $\mathfrak{m}(\mu_\ell^1)$ using W_1 and $\mathfrak{m}(f_*\mu_\ell^1)$ using W_2 to readily obtain the equality claimed in the lemma. To see that ϕ^1 has conjugate points if and only if ϕ^2 does we use [9][Theorem 4.4] which says that the asymptotic Maslov index of an invariant probability measure (with zero asymptotic cycle) is positive if and only if there are conjugate points in its support.

□

Let us now prove Theorem A. We know that the horocycle flow of a closed hyperbolic surface has $\mathfrak{a}(\mu_\ell) = \mathfrak{m}(\mu_\ell) = 0$. Since the horocycle flow is transitive, by Lemmas 4.1 and 4.2 the magnetic flow of (g,Ω) also has $\mathfrak{a}(\mu_\ell) = \mathfrak{m}(\mu_\ell) = 0$. The Proposition tells us that g has constant curvature k and $\Omega = \lambda\Omega_a$ with $k + \lambda^2 = 0$. If we let $a := -k$, then ag has curvature -1 and we have the situation of two closed hyperbolic surfaces with C^1-conjugate horocycle flows. The work of Marcus or Ratner [30, 34], tell us that ag and $\bar{g}$ must in fact be isometric as desired.

5. Closed Orbits in nontrivial free homotopy classes

We consider magnetic flows defined on an arbitrary closed connected manifold M. Let g be a Riemannian metric and let Ω be a closed 2-form. We will assume that Ω is *weakly exact*, that is, the lift of Ω to $\widetilde{M}$, the universal covering of M, is exact. Let θ be a primitive and let $c = c(g,\Omega)$ be Mañé's critical value, defined as in the Introduction. Recall that c is finite if and only if there is a bounded primitive. If θ is a bounded primitive, then our Lagrangian L satisfies all the hypotheses of Aubry-Mather theory for non compact manifolds as described for example in [6, 15, 16]. Recall that the *energy* in this case is simply the real valued function on $T\widetilde{M}$ given by $(x,v) \mapsto \frac{1}{2}|v|_x^2$.

We consider $\pi_1(M)$ acting on $\widetilde{M}$ by covering tranformations and we let $\Pi : \widetilde{M} \to M$ be the covering projection. Given a non-trivial element $\varphi \in \pi_1(M)$, let $Z_\varphi := \{\rho \in \pi_1(M) : \ \rho^{-1}\varphi\rho = \varphi\}$ be the centralizer of φ.

Theorem 5.1. *Let $k > c$ be given. Suppose that there exists a primitive θ which is Z_φ-invariant and for which*

$$\sup_{x\in\widetilde{M}} \frac{1}{2}\left|\theta_x\right|^2 \leq k - \varepsilon$$

for some $\varepsilon > 0$. Then the non-trivial free homotopy class determined by φ contains a closed magnetic geodesic with energy k.

Proof. Let θ be a Z_φ-invariant primitive with

$$\sup_{x\in\widetilde{M}}\frac{1}{2}\left|\theta_x\right|^2\leq k-\varepsilon$$

for some $\varepsilon>0$. If we consider the Lagrangian on $\widetilde{M}$ given by

$$L(x,v)=\frac{1}{2}|v|_x^2-\theta_x(v)$$

then

$$L(x,v)+k\geq\varepsilon>0 \tag{12}$$

for all $(x,v)\in TM$. Let us consider *Mañé's action potential*, which is given by

$$\Phi_k(x,y)=\inf_{T>0}\Phi_k(x,y;T)$$

where

$$\Phi_k(x,y;T):=\inf_{\gamma}A_{L+k}(\gamma),$$

and γ ranges among all absolutely continuous curves defined on $[0,T]$ connecting x to y. The potential Φ_k is a Lipschitz function which satisfies a triangle inequality

$$\Phi_k(x,y)\leq\Phi_k(x,z)+\Phi_k(z,y).$$

Note that the action potential is Z_φ-invariant, since θ is Z_φ-invariant.

Let $\psi\in\pi_1(M)$ be an arbitrary covering transformation. Since $\psi^*\theta-\theta$ is closed, there exists a smooth function $f_\psi:\widetilde{M}\to\mathbb{R}$ such that $\psi^*\theta-\theta=df_\psi$. The function f_ψ is uniquely defined up to addition of a constant, so from now on we shall assume that f_ψ is the unique function for which $f_\psi(x_0)=0$ where x_0 is some fixed point in $\widetilde{M}$. Note that from the definition of Φ_k we have:

$$\Phi_k(\psi x,\psi y)=\Phi_k(x,y)+f_\psi(y)-f_\psi(x) \tag{13}$$

for all $x,y\in\widetilde{M}$ and all $\psi\in\pi_1(M)$.

Lemma 5.2. *If $\psi_1^{-1}\varphi\psi_1=\psi_2^{-1}\varphi\psi_2$, then $f_{\psi_1}=f_{\psi_2}$.*

Proof. Clearly $\tau:=\psi_1\psi_2^{-1}\in Z_\varphi$. Hence $\psi_1^*\theta-\theta=\psi_2^*\tau^*\theta-\theta=\psi_2^*\theta-\theta$ which implies $df_{\psi_1}=df_{\psi_2}$. Thus $f_{\psi_1}=f_{\psi_2}$ since they both vanish at x_0.

□

A theorem due to Mañé [28, 8] ensures that given two distinct points x and y in $\widetilde{M}$ there exists a magnetic geodesic $\gamma:[0,R]\to\widetilde{M}$ with energy k, which connects x to y and realizes the potential, i.e.,

$$A_{L+k}(\gamma)=\Phi_k(x,y).$$

On account of (12)

$$\Phi_k(x,y)\geq\varepsilon\,R\geq\frac{\varepsilon}{\sqrt{2k}}\,d(x,y). \tag{14}$$

Let $a := \inf_{x\in\widetilde{M}} \Phi_k(x, \varphi x)$. Take a sequence of points x_n such that $\Phi_k(x_n, \varphi x_n) \to a$. Let K be a compact fundamental domain for the action of $\pi_1(M)$ on $\widetilde{M}$ and let $\psi_n \in \pi_1(M)$ be such that $\psi_n^{-1} x_n \in K$. Let $y_n := \psi_n^{-1} x_n$. Without loss of generality we can assume that y_n converges to some point $y \in K$. Using the triangle inequality for Φ_k we have:

$$\Phi_k(\psi_n y, \varphi\psi_n y) \le \Phi_k(\psi_n y, \psi_n y_n) + \Phi_k(\psi_n y_n, \varphi\psi_n y_n) + \Phi_k(\varphi\psi_n y_n, \varphi\psi_n y).$$

Using the φ-invariance of Φ_k and (13) we obtain

$$\begin{aligned}\Phi_k(\psi_n y,&\varphi\psi_n y)\\ &\le \Phi_k(y, y_n) + f_{\psi_n}(y_n) - f_{\psi_n}(y) + \Phi_k(y_n, y) + f_{\psi_n}(y) - f_{\psi_n}(y_n) + \Phi_k(x_n, \varphi x_n)\\ &= \Phi_k(y, y_n) + \Phi_k(y_n, y) + \Phi_k(x_n, \varphi x_n).\end{aligned}$$

But the expression $\Phi_k(y, y_n) + \Phi_k(y_n, y) + \Phi_k(x_n, \varphi x_n)$ is bounded in n, hence there exists $C > 0$ such that

$$\Phi_k(\psi_n y, \varphi\psi_n y) < C$$

for all n. Inequality (14) now implies that there exist only finitely many different elements of the form $\psi_n^{-1}\varphi\psi_n$. Hence for infinitely many values of n, $\psi_n^{-1}\varphi\psi_n$ equals some fixed covering tranformation, let us say, λ. Without loss of generality we shall assume that $\psi_n^{-1}\varphi\psi_n = \lambda$ for all n. Lemma 5.2 tells us that f_{ψ_n} is independent of n, so let us set $f_0 := f_{\psi_n}$.

Using (13) again we have

$$\Phi_k(\psi_n y, \varphi\psi_n y) = \Phi_k(y, \lambda y) + f_0(\lambda y) - f_0(y)$$

and

$$\Phi_k(x_n, \varphi x_n) = \Phi_k(\psi_n y_n, \varphi\psi_n y_n) = \Phi_k(y_n, \lambda y_n) + f_0(\lambda y_n) - f_0(y_n).$$

Since $\Phi_k(x_n, \varphi x_n) \to a$ we conclude that $\Phi_k(\psi_n y, \varphi\psi_n y) = a$ for all n, hence the points $\psi_n y$ realize the infimum of the function $x \mapsto \Phi_k(x, \varphi x)$.

Let z be one of these points, i.e. $\Phi_k(z, \varphi z) = a$. Consider the minimizing magnetic geodesic γ with energy k given by Mañé's theorem which connects z to φz and for which

$$A_{L+k}(\gamma) = \Phi_k(z, \varphi z).$$

We claim that $d\varphi(\dot\gamma(0)) = \dot\gamma(R)$. This implies that the projection of γ to M gives a closed magnetic geodesic in the free homotopy class determined by φ. To prove that $d\varphi(\dot\gamma(0)) = \dot\gamma(R)$ we play the same game as in Riemannian geometry. Consider $b > 0$ small and note that

$$\begin{aligned}\Phi_k(\gamma(b), \varphi\gamma(b)) &\le \Phi_k(\gamma(b), \varphi z) + \Phi_k(\varphi z, \varphi\gamma(b))\\ &= \Phi_k(\gamma(b), \varphi z) + \Phi_k(z, \gamma(b))\\ &= \Phi_k(z, \varphi z)\end{aligned}$$

where in the first equality we used the φ-invariance of Φ_k. Since $x \mapsto \Phi_k(x, \varphi x)$ achieves its minimum at z, we must have $d\varphi(\dot\gamma(0)) = \dot\gamma(R)$ which concludes the proof of the theorem.

□

The next lemma will be important for us. Its proof is a fairly standard application of amenability.

Lemma 5.3. *Let $\Gamma \subset \pi_1(M)$ be an amenable subgroup. For any $k > c$, there exists a smooth Γ-invariant primitive ϑ such that*

$$\sup_{x\in\widetilde{M}} \frac{1}{2}\left|\vartheta_x\right|^2 \leq k.$$

Proof. Since Γ is amenable it has a right invariant mean on $\ell^\infty(\Gamma)$, that is, there exists a bounded linear functional $m : \ell^\infty(\Gamma) \to \mathbb{R}$ such that

(1) $m(a) = a$ for a constant function a;
(2) $m(a_1) \geq m(a_2)$, if $a_1(\varphi) \geq a_2(\varphi)$ for all $\varphi \in \Gamma$;
(3) $m(\varphi_* a) = m(a)$, where $\varphi_* a(\psi) := a(\psi\varphi)$.

By the definition of Mañé's critical value, given $k > c$, there exists a primitive θ such that

$$\sup_{x\in\widetilde{M}} \frac{1}{2}\left|\theta_x\right|^2 \leq k. \tag{15}$$

Given $(x, v) \in TM$, consider that function $a_{(x,v)} : \Gamma \to \mathbb{R}$ given by $a_{(x,v)}(\varphi) = \theta_{\varphi(x)}(d\varphi(v))$. Since φ acts by isometries, inequality (15) implies that $a_{(x,v)} \in \ell^\infty(\Gamma)$. Hence we can set:

$$\vartheta_x(v) := m(a_{(x,v)}).$$

The linearity and continuity of m implies that ϑ is a smooth 1-form and by property (3), ϑ is Γ-invariant. Moreover if γ is any closed curve, the linearity and continuity of m imply

$$\int_\gamma \vartheta = m\left(\varphi \mapsto \int_\gamma \varphi^*\theta\right). \tag{16}$$

But $\int_\gamma \varphi^*\theta$ is independent of φ. This can be seen as follows. Since $\widetilde{M}$ is simply connected, there exists a smooth map $F : \mathbb{D} \to \widetilde{M}$, where $\mathbb{D}$ is a 2-disk and the restriction of F to $\partial\mathbb{D}$ is γ. Hence

$$\int_\gamma \varphi^*\theta = \int_{\partial\mathbb{D}} F^*\varphi^*\theta = \int_{\mathbb{D}} F^*\varphi^* d\theta.$$

But $d\theta = \widetilde{\Omega}$ and hence is Γ-invariant, so the last integral is independent of φ. Thus from (16)

$$\int_\gamma \vartheta = \int_\gamma \theta$$

for any closed curve γ. This implies that $\vartheta - \theta$ is an exact form.

Finally, since $a_{(x,v)}(\varphi) \leq \sqrt{2k}$ for all $x \in \widetilde{M}$, all $v \in T_x\widetilde{M}$ with norm one and all $\varphi \in \Gamma$, property (2) implies that $\vartheta_x(v) \leq \sqrt{2k}$ for all $x \in \widetilde{M}$ and all $v \in T_x\widetilde{M}$ with

norm one. Thus

$$\sup_{x\in\widetilde{M}} \frac{1}{2}\,|\vartheta_x|^2 \le k$$

as desired.

□

Corollary 5.4. *Suppose that $\pi_1(M)$ is amenable and Ω is not exact. Then $c(g,\Omega) = \infty$.*

Proof. If $c(g,\Omega)$ is finite, $\widetilde{\Omega}$ admits a bounded primitive and by the previous lemma, $\widetilde{\Omega}$ admits a $\pi_1(M)$-invariant primitive θ. The form θ descends to M showing that Ω is exact. □

Theorem 5.5. *Let $k > c$ and let $\varphi \in \pi_1(M)$ be a non-trivial element with amenable centralizer. Then the non-trivial free homotopy class determined by φ contains a closed magnetic geodesic with energy k.*

Proof. It follows right away from Theorem 5.1 and Lemma 5.3. □

Remark 5.6. If $\pi_1(M)$ is the fundamental group of a closed manifold of negative curvature and φ is non-trivial, Preissman's theorem implies that Z_φ coincides with the infinite cyclic group generated by φ, which is of course amenable. Thus we can apply Theorem 5.5 to any non-trivial free homotopy class. In the next section we will apply the theorem to a closed surface of genus ≥ 2.

5.1. **Proof of Theorem C.** We will need the following result which was proven by G. Contreras [7] in the exact case and extended by O. Osuna [31] to the weakly exact case as part of his Ph.D thesis work.

Theorem 5.7. *For almost every k in the interval $(0, c)$ there exists a closed contractible magnetic geodesic with energy k.*

The first item in Theorem C follows from Corollary 5.4 and Theorem 5.7. The second item in Theorem C follows from Theorem 5.5 and Theorem 5.7.

Theorem 5.7 is proved by showing that an appropriate action functional (which depends on the energy level) on the space of contractible loops exhibits a mountain pass geometry. Then standard Morse theory gives the existence of critical points whenever the Palais-Smale condition holds. It has been known for some time, that the Palais-Smale condition can only fail in the "time direction". An argument originally due to M. Struwe can now be applied to the mountain pass geometry to overcome this difficulty for almost every energy level.

6. Proof of Theorem B

In this section we return to the case in which M is a closed surface of genus ≥ 2. The following lemma has independent interest.

Lemma 6.1. *Suppose the magnetic flow of (g,Ω) is Mañé critical. If there exists a nontrivial free homotopy class without closed magnetic geodesics, then there exists*

an invariant Borel probability measure ν with $\mathfrak{a}(\nu) = \mathcal{S}(\nu) = 0$. Equivalently, the magnetic flow is not of contact type.

Proof. Let σ be the nontrivial free homotopy class without closed magnetic geodesics. Suppose that σ is generated by the covering transformation φ. As in the proof of Theorem 5.5 we consider a φ-invariant action potential $\Phi_k \geq 0$ for all $k > c = 1/2$. Now take a decreasing sequence k_n approaching $1/2$ as $n \to \infty$. Theorem 5.5 gives points x_n and orbits $\gamma_n : [0, T_n] \to \widetilde{M}$ with energy k_n connecting x_n and φx_n. The orbits γ_n project to M as closed orbits in the class σ and

$$0 \leq A_{L_n+k_n}(\gamma_n) = \Phi_{k_n}(x_n, \varphi x_n).$$

Moreover, x_n is a minimum of $x \mapsto \Phi_{k_n}(x, \varphi x)$. Hence if y is any point in $\widetilde{M}$,

$$\Phi_{k_n}(x_n, \varphi x_n) \leq \Phi_{k_n}(y, \varphi y) \leq C$$

for some constant $C > 0$, since $L_n(x, v) + k_n \leq 1/2 + k_n + \sqrt{2k_n}$ for all $(x, v) \in TM$ with $|v|_x \leq 1$. Hence

$$0 \leq A_{L_n+k_n}(\gamma_n) \leq C. \tag{17}$$

We now observe that $\inf_n T_n > 0$, otherwise we would get curves in the class σ with arbitrarily short lengths, which is impossible. If $\sup_n T_n < \infty$, by passing to a subsequence if necessary, we can assume that $T_n \to T_0$ and that the projection of $(\gamma_n(0), \dot{\gamma}_n(0))$ to TM converges to some point $(p, v) \in SM$. The orbit of (p, v) gives rise to a closed magnetic geodesic with period T_0 in the homotopy class σ. Since we are assuming that σ has no such orbits we must have $\sup_n T_n = \infty$. Without loss of generality we shall assume from now on that $T_n \to \infty$.

Let us indicate with a tilde the lift of objects on M (or SM) to $\widetilde{M}$ (or $S\widetilde{M}$). Note that

$$(L + k_n)(\gamma_n, \dot{\gamma}_n) = 2k_n - \theta_n(\dot{\gamma}_n) = (\widetilde{\alpha} - \widetilde{\pi}^*\theta_n)(\widetilde{X})(\gamma_n, \dot{\gamma}_n). \tag{18}$$

Let Θ be a primitive of ω in a neighbourhood of SM. Since

$$d(\widetilde{\alpha} - \widetilde{\pi}^*\theta_n) = -\widetilde{\omega} = -d\widetilde{\Theta}$$

there exists a smooth closed 1-form ρ_n defined in a neighbourhood of $S\widetilde{M}$ for which

$$\widetilde{\alpha} - \widetilde{\pi}^*\theta_n = -\widetilde{\Theta} + \rho_n. \tag{19}$$

Combining (18) and (19) we obtain:

$$(L + k_n)(\gamma_n, \dot{\gamma}_n) = -\widetilde{\Theta}(\widetilde{X})(\gamma_n, \dot{\gamma}_n) + \rho_n(\widetilde{X})(\gamma_n, \dot{\gamma}_n). \tag{20}$$

Let ν_n be the Borel probability measures on TM given by:

$$\int f \, d\nu_n := \frac{1}{T_n} \int_0^{T_n} f(\Pi(\gamma_n(t)), d\Pi(\dot{\gamma}_n(t))) \, dt.$$

Without loss of generality we can assume that ν_n converges weakly to an invariant measure ν. Since $k_n \to 1/2$, the measure ν has support in SM. Let us check that

$\mathfrak{a}(\nu) = \mathcal{S}(\nu) = 0$. Using (17) and (20) we see that

$$0 = \lim \frac{1}{T_n} A_{L+k_n}(\gamma_n) = -\int \Theta(X)\, d\nu + \lim \frac{1}{T_n} \int_{(\gamma_n,\dot{\gamma}_n)} \rho_n$$

and therefore to show that $\mathfrak{a}(\nu) = 0$ it suffices to check that

$$\lim \frac{1}{T_n} \int_{(\gamma_n,\dot{\gamma}_n)} \rho_n = 0.$$

Equality (19) implies that ρ_n is φ-invariant and its norm is bounded by a constant, let us say A, independent of n. Let $\widehat{M}$ be the manifold obtained by taking the quotient of $\widetilde{M}$ by the action of the cyclic group generated by φ. The curves γ_n project to *simple* closed curves in $\widehat{M}$ which are all homotopic and therefore the curves $(\gamma_n, \dot{\gamma}_n)$ in $T\widetilde{M}$ project to closed curves Γ_n in a neighbourhood of $S\widehat{M}$, whose homology class $[\Gamma_n]$ is independent of n. The form ρ_n descends to a closed 1-form $\widehat{\rho}_n$ defined in a neighbourhood of $S\widehat{M}$. Observe that

$$\int_{(\gamma_n,\dot{\gamma}_n)} \rho_n = \int_{\Gamma_n} \widehat{\rho}_n = \langle [\widehat{\rho}_n], [\Gamma_n] \rangle.$$

Since $[\Gamma_n]$ is independent of n and $\widehat{\rho}_n$ is bounded by A we have

$$\lim \frac{1}{T_n} \langle [\widehat{\rho}_n], [\Gamma_n] \rangle = 0$$

as desired.

Let us prove that $\mathcal{S}(\nu) = 0$. Let Υ be any closed 1-form on SM. Since $\pi^* : H^1(SM, \mathbb{R}) \to H^1(M, \mathbb{R})$ is an isomorphism, there exists a closed 1-form δ on M and a smooth function G on SM such that $\Upsilon = \pi^*\delta + dG$. Thus

$$\int \Upsilon(X)\, d\nu = \int \pi^*\delta(X)\, d\nu$$

and to prove that $\mathcal{S}(\nu) = 0$ it suffices to show that

$$\int \pi^*\delta(X)\, d\nu = 0.$$

But

$$\int \pi^*\delta(X)\, d\nu = \lim \frac{1}{T_n} \int_{\Pi\circ\gamma_n} \delta = \lim \frac{1}{T_n} \langle [\delta], [\Pi \circ \gamma_n] \rangle.$$

The curves $\Pi \circ \gamma_n$ are all in the same free homotopy class σ, hence the homology class $[\Pi \circ \gamma_n]$ is independent of n which gives $\mathcal{S}(\nu) = 0$ as desired.

To complete the proof of the lemma, recall that SM is of contact type if and only if for all invariant Borel probability measures μ with zero asymptotic cycle, $\mathfrak{a}(\mu) \neq 0$ (cf. [12][Proposition 2.4]).

$\square$

Let us prove Theorem B. The first observation is that we always have semistatic curves starting at any point in $\widetilde{M}$ [11, 6, 16]. These are magnetic geodesics $\gamma : [0,\infty) \to \widetilde{M}$ such that

$$A_{L+1/2}(\gamma|_{[s,t]}) = \Phi_{1/2}(\gamma(s),\gamma(t))$$

for $0 \leq s < t < \infty$ where as before

$$\Phi_{1/2}(x,y) = \inf_{T>0} \Phi_{1/2}(x,y;T).$$

A semistatic curve must be free of conjugate points in $[0,\infty)$ (see [10][Corollary 4.2]) and hence the ω-limit set of the projection of γ to M must also be free of conjugate points. Since we are assuming that the magnetic flow is uniquely ergodic, this implies that all SM is free of conjugate points. On account of Lemma 6.1 and unique ergodicity, $\mathfrak{a}(\mu_\ell) = 0$ and the theorem follows from the Proposition.

Remark 6.2. We can rephrase Lemma 6.1 by saying that if a Mañé critical magnetic flow is of contact type, then every nontrivial free homotopy class contains a closed magnetic geodesic. Since there are nontrivial free homotopy classes with the property that any closed curve in them is homologous to zero, we obtain, in particular, closed magnetic geodesics homologous to zero. Recall that the *Weinstein conjecture* says that every Reeb vector field on a closed 3-manifold admits a closed orbit. The *strong Weinstein conjecture* asserts that in fact one can find finitely many closed orbits which form a cycle homologous to zero. There has been lots of progress recently regarding this conjecture (cf. [1]). However, the work of J. Entyre [14] implies that magnetic flows, with Ω symplectic, are *excluded* from all the known cases in which the conjecture holds.

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